Physics 218C Week 4 Lecture Note (Apr. 19th, 21st)

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Contents

1 Review From Last Week

Last time, we investigated the collisional drift-wave (DW) instability model—Hasegawa-Wakatani (H-W) equations:

$$\begin{cases} \rho_s^2 \dfrac{\partial}{\partial t}\nabla_\perp^2 \widetilde{\phi} + \dfrac{\widetilde{v}_r}{n_0}\dfrac{\partial \rho_s^2 \langle \nabla_\perp^2 \phi \rangle}{\partial r} = D_\parallel \nabla_\parallel^2 (\widetilde{\phi} - \dfrac{T}{|e|}\dfrac{\widetilde{n}}{n_0}) + \nu \nabla^2 \rho_s^2 \nabla_\perp^2 \widetilde{\phi} \\ \\ \dfrac{\partial}{\partial t}\dfrac{\widetilde{n}}{n_0} + \dfrac{\widetilde{v}_r}{n_0}\dfrac{\partial \langle n \rangle}{\partial r} = D_\parallel \nabla_\parallel^2 (\widetilde{\phi} - \dfrac{T}{|e|}\dfrac{\widetilde{n}}{n_0}) + \dfrac{D_0}{n_0}\nabla_\perp^2 \widetilde{n}, \end{cases} \tag{1}$$

where ϕ is electrical potential, ρ_s is gyro-radius, $D_\parallel$ is parallel diffusion, ν is kinematic viscosity, and D_0 mass diffusivity. We defined α as $\frac{k_\parallel^2 v_{th,e}^2/\gamma_{ie}}{\omega}$, where γ_{ie} is electron-ion collision rate. Usually, we have adiabatic limit, or DW regime, with adiabatic parameter $\alpha > 1$. This regime (DW instability has growth and reach relaxation) requires $\omega < \omega_*(diamagnetic\ frequency)$ and under the assumption of a non-Boltzmann correction $\dfrac{\widetilde{n}}{n_0} \simeq \dfrac{|e|\widetilde{\phi}}{T} + \widetilde{h}$, where $\widetilde{h}$ is due to non-adiabatic electron. In this regime, we have several phenomenon/properties:

- non-zero $\langle \widetilde{v}_r \widetilde{n} \rangle$, due to non-zero parallel dissipation and $\omega < \omega_*$
- system allow outward radial flux and instability (criterion of $\omega < \omega_*$)
- energy gained from gradient relaxation exceeds the cost of "pumping" the potential

When $\alpha \to \infty$, H-W model will reduced to Hasegawa-Mima (H-M) model—that describes *collissionless* DW instability. The H-M equation is

$$\frac{\partial}{\partial t}(\nabla^2\phi - \phi) - [(\nabla\phi \times \hat{z}) \cdot \nabla][\nabla^2\phi - \ln(n_0)] = 0, \tag{2}$$

and potential vorticity (PV) is conserved, where $PV \equiv \phi - \rho_s^2 \nabla^2 \phi + \ln(n_0)$. To differentiate the DW instability ($\alpha > 1$) and ideal/resistive MHD ($\alpha < 1$), the key is understanding the Ohm's Law balance:

$$(\mathbf{E} + \mathbf{v} \times \mathbf{B})_\parallel + \underbrace{T\nabla_\parallel n}_{DW \text{ trigger}} = \underbrace{\eta \mathbf{J}_\parallel}_{\text{dissipation}} . \tag{3}$$

Note that though we discuss how ∇n trigger DW instability, ∇T can also do so. And that an H-W system can be describe by the **Schmidt number**:

$$Sc \equiv \frac{\nu}{D_0} = \frac{\text{kinematic viscosity}}{\text{mass diffusivity}}, \tag{4}$$

that illustrates the ratio of the last term in momentum and density equation.

2 Zonal mode

Decompose total density we *measure* into the mean and the perturbation such that

$$\langle n \rangle \equiv n_{0(r)} + \delta n_{(r)},$$

where perturbation $\delta n_{(r)}$ can be viewed as "corrugation" upon the mean, and it is due to the feedback (i.e. transport) of fluctuations on profile (see Fig. 1). Here after we replace notation of perturbation of density $\widetilde{n}/n_0$ in Eq.1 as δn, and we call $\delta n_{(r)}$ the **zonal density perturbation**. Similarly, $\nabla_r^2 \phi_{(r)}$ the **zonal vorticity**, which is from divergence of polarization current $\nabla \cdot \mathbf{J}_{pol} = -\dfrac{\partial}{\partial t}\rho_{pol}$ and under the assumption of azimuthal symmetry. Here, ϕ is from the $\mathbf{E} \times \mathbf{B}$ zonal flow. It is important to clarify that

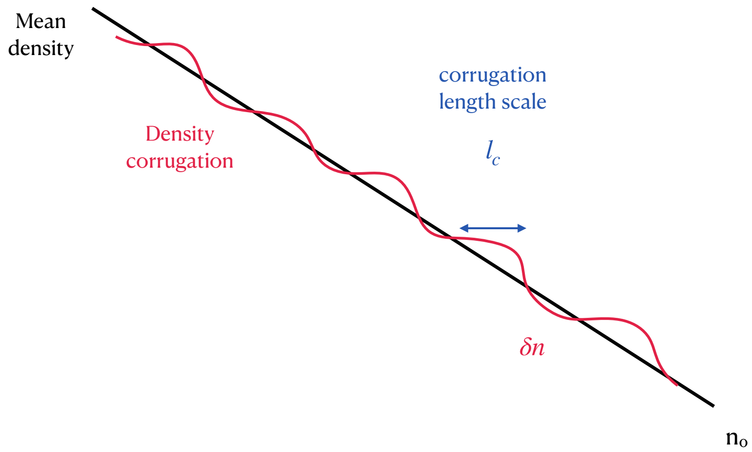

Figure 1: Mean density and density corrugation (zonal density). Zonal density feeds back on the density profile, which evolves the zonal mode

- $E \times B$ flow: is a zonal flow move in $v_{E\times B}$ velocity.
- mass flow: $v_{mass} = \int d^3 v \mathbf{v} f$, where f is distribution function of velocity.

Homework: Work out the mass flow for fluid ions. Keeping $E \times B$, diamagnetic, polarization.

Zonal flow arises naturally via vorticity $\nabla^2\phi$. Zonal flow produce $E \times B$ shear, hence the operator $\frac{d}{dt}$ now has shear flow component in cartesian coordinate (where x in radial, y in poloidal, and z in toroidal direction) such that

$$\frac{d}{dt} \to \frac{\partial}{\partial t} + v_{E\times B(r)} \frac{\partial}{\partial y} + \mathbf{v} \cdot \nabla, \tag{5}$$

where $v_{E\times B(r)}$ is $E \times B$ shear flow and have r-dependence due to $\phi = \phi_{(r)}$. And the zonal mean

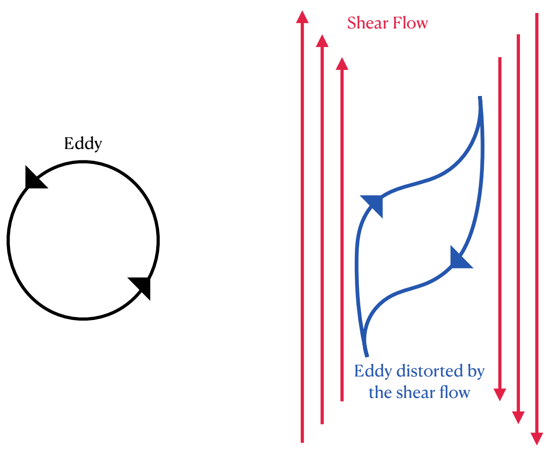

Figure 2: Shear flows disorder and rip apart the eddies.

shear is equal to the mean vorticity:

$$\frac{\partial}{\partial r}\langle v_{E\times B(r)}\rangle = \langle \partial_r^2 \phi \rangle. \tag{6}$$

Hence, we have the total vorticity mean

$$\langle \nabla_\perp^2 \phi \rangle = \underbrace{v_{E\times B(r),0}}_{\substack{\text{large scale}\\ \text{shear flow}}} + \underbrace{\langle \partial_r^2 \phi_{(r)} \rangle}_{\substack{\text{generate from}\\ \text{micro-scale DW}}} . \tag{7}$$

Notice that the shear flow can be *multi-scaled*. Shear flow can be decomposed into several terms based on scales

- macro-scale: $v_{E\times B(r),0}$ that is prescribe, and
- meso-scale (zonal-flow scale): $\langle \partial_r^2 \phi_{(r)} \rangle$ generate from the *micro-scale* DW.

2.1 Zonal Density evolution

Now, we are interested in **zonal density evolution**. This can be derived from average over the density H-W equation (see Eq. 1):

$$\frac{\partial}{\partial t}\langle \widetilde{n} \rangle + \frac{\partial}{\partial r}\langle \widetilde{v}_r \widetilde{n} \rangle = \underbrace{S_n}_{\text{source}} + \underbrace{D_0 \nabla_r^2 \langle n \rangle}_{\substack{\text{ambient}\\ \text{diffusion}}}, \tag{8}$$

where the angle braket is the zonal ensemble average $\langle\rangle \equiv \frac{1}{T}\int dt \frac{1}{L_y}\int dy$, $\langle \widetilde{v}_r \widetilde{n} \rangle = \langle \widetilde{v}_r \widetilde{h} \rangle$ is radial particle flux due *only* to the *non-adiabatic* part because the Boltzmann piece cancels, and S_n is the particle source. So, **the particle density flux evolves the zonal density and the density profile**. The key question then will be how to calculate the particle density flux $\langle \widetilde{v}_r \widetilde{h} \rangle$? One way is from quasi-linear (QL) theory and get the linear response of the non-adiabatic density to the perturbation flow $\widetilde{v}_r$. Notice that the QL theory requires system has Kubo number $Ku < 1$, and it is good for the system has dissipative dynamics (i.e. $\widetilde{h} = \frac{(\omega-\omega_*)}{-i\cancel{\omega}+\frac{k_\parallel^2 v_{th,e}^2}{\gamma_{ie}}}$). Notice that $(\omega-\omega_*)$ will induce a non-linear frequency shift. BTW, the corrugation $\delta n_{(r)}$ will modify the ω_* since $\omega_* \propto \nabla\langle n \rangle$, and mean density will be modified by the corrugation. The scale of density corrugation l_c lies on the mesoscale (see Fig.1 and 3), and can be predicted as a geometric mean of length of mean density and the gryro-radius $l_c \simeq \sqrt{\rho_s L_n}$. And there is also a spectrum of zonal flow on mesoscale. Hence, QL type thinking will be that **zonal density (corrugation) feeds back on the density profile, which evolves the zonal mode**. And from QL theory, we have

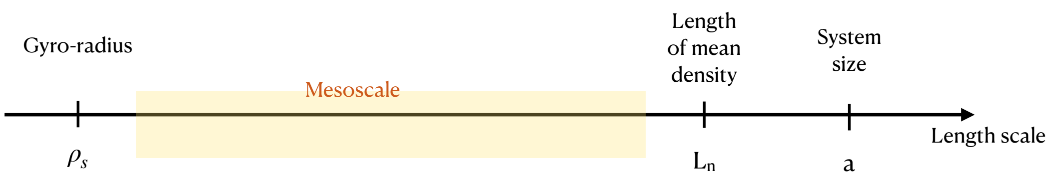

Figure 3: Lengthscale

$$\frac{\partial}{\partial t}\delta n = \frac{\partial}{\partial r} D_n \frac{\partial}{\partial r}\delta_n, \tag{9}$$

where density diffusivity D_n (where $\omega \ll k_\parallel^2 v_{e,th}^2/\gamma_{ie}$) is

$$D_n \simeq \sum_k |\widetilde{v}_{r,k}|^2 \frac{k_\perp^2 \rho_s^2}{1+k_\perp^2\rho_s^2} \frac{\omega_k \gamma_{ei}}{k_\parallel^2 v_{e,th}^2} \simeq \sum_k |\widetilde{v}_{r,k}|^2 \frac{k_\perp^2\rho_s^2}{1+k_\perp^2\rho_s^2}\frac{1}{\alpha}, \tag{10}$$

where ion-polarization drift $k_\perp^2\rho_s^2$ comes from the $(\omega - \omega_*)$. Notice that the term $\dfrac{1}{\alpha}$ comes from the collision, origin of *irreversibility*, and causes dissipative phase shift.
Recall the QL equation for the 1D Vlasov equation:

$$\frac{\partial}{\partial t}\langle f\rangle = \frac{\partial}{\partial v} D_v \frac{\partial}{\partial v}\langle f\rangle, \tag{11}$$

where D_v is QL diffusion in velocity

$$D_v = \sum_k \frac{q^2}{m^2}|E_k|^2 \frac{\gamma_k}{(\omega - kv)^2 + |\gamma_k|^2}, \tag{12}$$

where E_k is the scattering fields.
Recall that the term $\dfrac{\gamma_k}{(\omega-kv)^2+|\gamma|^2}$ comes from the **wave-particle resonance** and causes a **resonant phase shift**. Here, we don't have collision, so the irreversibility comes from the chaos (Chirikov, 1969). More can be found in **KAM theorem** (Kolmogorov, 1954; Möser, 1962).

Homework: (a) Derive the QL equation for the collissionless drift wave.

Hint: you will have similar equations as in Eq. 11 and 12 .

2.2 Vorticity Flux and Enstrophy

We can do a similar QL analysis for the evolution of the vorticity (or polarization charge)

$$\frac{\partial}{\partial t}\langle \nabla_r^2 \phi\rangle + \frac{\partial}{\partial r}\underbrace{\langle \widetilde{v}_r \nabla_r^2 \widetilde{\phi}\rangle}_{\text{flux of vorticity}} = \underbrace{\nu \nabla_r^2 \langle \nabla_r^2 \phi\rangle}_{\text{viscous diffusion}} . \tag{13}$$

The key to the zonal flow evolution is the **flux of vorticity** and it requires some discussion. **What is the physics of vorticity flux?** First, we have a correlator of $\widetilde{v}_r$ with $\nabla_r^2\widetilde{\phi}$

$$\begin{aligned}\langle \widetilde{v}_r \nabla_r^2 \widetilde{\phi}\rangle &= \langle \frac{\partial}{\partial y}\widetilde{\phi}(\frac{\partial^2}{\partial x^2}\widetilde{\phi} + \frac{\partial^2}{\partial y^2}\widetilde{\phi})\rangle & (14)\\ &= \langle \frac{\partial}{\partial y}\widetilde{\phi}(\frac{\partial^2}{\partial x^2}\widetilde{\phi})\rangle, & (15)\end{aligned}$$

where the second term has odd term ∂_y^3 and after ensemble average it becomes zero, under the assumption that the **poloidal and toroidal symmetry** (but here we ignore $k_z \to 0$). So we have

$$\begin{aligned}\langle \widetilde{v}_r \nabla_r^2 \widetilde{\phi}\rangle &= \langle \frac{\partial}{\partial x}(\frac{\partial}{\partial x}\widetilde{\phi} + \frac{\partial}{\partial y}\widetilde{\phi})\rangle - \langle \overset{\text{odd in } k_y}{\cancel{\frac{\partial^2}{\partial xy}\widetilde{\phi}\frac{\partial}{\partial x}}}\widetilde{\phi}\rangle & (16)\\ &= \frac{\partial}{\partial x}\langle \frac{\partial}{\partial x}\widetilde{\phi} + \frac{\partial}{\partial y}\widetilde{\phi}\rangle & (17)\\ &= \frac{\partial}{\partial x}\langle \widetilde{v}_r \widetilde{v}_\theta\rangle & (18)\end{aligned}$$

So we have

$$\langle \widetilde{v}_r \nabla_r^2 \widetilde{\phi} \rangle = \underbrace{\frac{\partial}{\partial x} \langle \widetilde{v}_r \widetilde{v}_\theta \rangle}_{\substack{E \times B \\ \text{Reynolds Force}}} \tag{19}$$

Hence, the physics of vorticity is that **the vorticity flux act as the Reynolds Force that drives the $E \times B$ shear flow**[1].

> *Homework: (a) Prove the magnetic Taylor Identity.*
> *Show $\langle \widetilde{B}_r \widetilde{J}_\parallel \rangle = ?$*
> *(b) Relate (a) to charge balance condition. Show*
>
> $$\frac{\partial}{\partial t} \langle \nabla_\perp^2 \widetilde{\phi} \rangle = \frac{\partial}{\partial r} [\underbrace{\langle \widetilde{v}_r \nabla_\perp^2 \widetilde{\phi} \rangle}_{E \times B \text{ Reynolds stress}} - \underbrace{\langle \widetilde{B}_r \widetilde{J}_\parallel \rangle}_{\text{Maxwell Stress}}], \tag{20}$$
>
> *and what is the meaning?*

2.3 Relate the vorticity and transport—PV and Enstrophy

How do we relate the vorticity $\nabla_\perp^2 \phi$ (see Sec.2.2) and transport δn (see Sec.2.1)? The **free energy for the DW instability is the density gradient** $\partial_x n_0$. In limited cases, vorticity gradient can drive the Kelvin-Helmholtz instability (will talk in later classes). A fair question we should ask is

> *The story we have now is on one hand, the density gradient provide the free energy (entropy production), drive turbulence, and hence the system reach its relation. On the other hand, these turbulence unset zonal flows. So questions will be—how we treat the zonal density and zonal vorticity on the same bases? Is there more underlying physics?*

To answer these questions, understanding PV is important. Hereafter we are working in a limit that the (collisional) kinematic viscosity equals (collisional) mass diffusivity $\nu = D_0$ (or $Sc = 1$). Hence, we can simplify H-W equations (see Eq.1) as

$$\frac{\partial}{\partial t}(\delta n - \rho_s^2 \nabla_\perp^2 \widetilde{\phi}) + \frac{\widetilde{v}_r}{n_0} \frac{\partial}{\partial r}(\langle n \rangle - \rho_s^2 \langle \nabla_\perp^2 \phi \rangle) = \nu \nabla_\perp^2 (\delta n - \rho_s^2 \nabla_\perp^2 \widetilde{\phi}). \tag{21}$$

The LHS looks like continuity equation. If $\nu \to 0$, we have

$$\boxed{\frac{d}{dt}(\delta n - \rho_s^2 \nabla_\perp^2 \widetilde{\phi}) = 0.}$$

Hence, we can define this conserved term as PV

$$PV \equiv \delta n - \rho_s^2 \nabla_\perp^2 \widetilde{\phi} \equiv q \tag{22}$$

$$\equiv \frac{|e| \widetilde{\phi}}{T} + \underbrace{\widetilde{h}}_{\substack{\text{non-} \\ \text{Boltzmann}}} - \rho_s^2 \nabla_\perp^2 \widetilde{\phi} \tag{23}$$

So we have PV conservation

$$\frac{d}{dt} PV = 0, \text{ or } \mathcal{O}_{(\nu)}. \tag{24}$$

This indicate that changing density leads to changing flow shear. To be specific:

[1] Wood & McIntyre (2010) propose an idea that once there is 1D symmetry in PV mixing, then you can prove there will be a zonal flow formation. And we can also **Taylor identity** (Taylor, 1915) that link the vorticity flux to Reynolds stress, and there a generalization form of Taylor identity called **Eliassen-Palm relation** (Eliassen, 1960) in geo-fluid dynamics.

- any relaxation of density must be accompanied by a vorticity flux, in order to conserve PV.
- any density relaxation drives the vorticity flux and hence creates the zonal flow.

Note that if we view PV as a "charge", then density corrugation δn can be viewed as a guiding center charge (i.e. electron) and $\rho_s^2 \nabla_\perp^2 \widetilde{\phi}$ can be viewed as a polarization charge (i.e. ion)

$$PV \equiv \underbrace{\delta n}_{\text{electron (guiding center charge)}} - \underbrace{\rho_s^2 \nabla_\perp^2 \widetilde{\phi}}_{\text{ion (polarization charge)}} ,$$

and **PV conservation is a conservation of total charge** .
One can define "mean-square charge" as **Potential Enstrophy** ($\mathcal{E} \equiv \langle q^2/2 \rangle$). The potential enstrophy is also a conserved quantity since if q is conserved (i.e. no dissipation), so is q^2. We times a q to LHS and RHS of Eq. 21, we have the *charge balance equation*:

$$\frac{\partial}{\partial t} E + \underbrace{\langle \widetilde{v}_r q \rangle \frac{\partial}{\partial r} \langle q \rangle}_{\text{PV flux production}} + \underbrace{\frac{\partial}{\partial r} \langle \widetilde{v}_r \mathcal{E} \rangle}_{\text{turb. spreading}} = \underbrace{\nu \nabla_\perp^2 (E)}_{\text{dissipation term}} . \tag{25}$$

Note that the third term on LHS is third-order of fluctuation, reflecting the **turbulenct spreading—the turbulent transport of potential enstropy ($\mathcal{E}$)**. By apply Taylor identity on PV flux production, we have

$$\langle \widetilde{v}_r q \rangle \frac{\partial}{\partial r} \langle q \rangle = \left[\langle \widetilde{v}_r \widetilde{h} \rangle - \frac{\partial}{\partial r} \langle \widetilde{v}_r \widetilde{v}_\theta \rangle \right] \frac{\partial}{\partial r} \langle q \rangle \tag{26}$$

Rewrite Eq.25, we have

$$\underbrace{\langle \widetilde{v}_r \widetilde{h} \rangle}_{\text{particle flux}} - \underbrace{\frac{\partial}{\partial r} \langle \widetilde{v}_r \widetilde{v}_\theta \rangle}_{\text{vorticity flux}} = \frac{-1}{\frac{\partial}{\partial r} \langle q \rangle} \left[\frac{\partial}{\partial t} \mathcal{E} + \underbrace{\frac{\partial}{\partial r} \langle \widetilde{v}_r \mathcal{E} \rangle}_{\text{turb. spreading}} - \underbrace{\nu \nabla_\perp^2 \mathcal{E}}_{\text{dissipation}} \right], \tag{27}$$

and it is important that $\frac{\partial}{\partial r} \langle q \rangle \nrightarrow 0$. If $\frac{\partial}{\partial r} \langle q \rangle = 0$, we have shear flow instability. This indicates that **in steady state ($\partial_t = 0$), the particle flux should be equal to the vorticity flux, up to higher order contribution or dissipation**. Namely

$$\underbrace{\langle \widetilde{v}_r \widetilde{h} \rangle}_{\text{particle flux}} \simeq \underbrace{\frac{\partial}{\partial r} \langle \widetilde{v}_r \widetilde{v}_\theta \rangle}_{\text{vorticity flux}} + \mathcal{O}_{(\text{spreading+ dissipation})} .$$

Hence, PV conservation relates the particle flux to the vorticity flux (zonal flow drive).
One can go further by considering drag of zonal flow and modifying the vorticity flux (Reynolds force) term as

$$-\frac{\partial}{\partial r} \langle \widetilde{v}_r \widetilde{v}_\theta \rangle = \frac{\partial}{\partial t} \langle v_{E \times B} \rangle + \mu_{\text{drag}} \langle v_{E \times B} \rangle .$$

Plugging the above modification to Eq.27, we have

$$\boxed{\frac{\partial}{\partial t} \left[\frac{\mathcal{E}}{\partial_r \langle q \rangle} + \langle v_{E \times B} \rangle \right] + \langle \widetilde{v}_r \widetilde{h} \rangle = + \frac{-1}{\frac{\partial}{\partial r} \langle q \rangle} \left[\underbrace{\frac{\partial}{\partial r} \langle \widetilde{v}_r \mathcal{E} \rangle}_{\text{turb. spreading}} - \underbrace{\nu \nabla_\perp^2 \mathcal{E}}_{\text{dissipation term}} \right] - \mu_{\text{drag}} \langle v_{E \times B} \rangle} \tag{28}$$

This is the **Charney-Drazin (C-D) Theorem** (Charney & Stern, 1962), which illustrates constrain of PV conservation on zonal flow production and its relation to transport. Here, the particle flux is turbulent drive and the drag term can drive zonal flow in steady state. The term $\frac{\mathcal{E}}{\partial_r \langle q \rangle}$ is **wave**

momentum density (WMD). And $k_\theta \dfrac{\mathcal{E}}{\partial_r \langle q \rangle}$ is pseudo-momentum density [2].

If there is no spreading, dissipation, drag, or particle flux, then Eq.28 becomes

$$\boxed{\frac{\partial}{\partial t}\left[\frac{\mathcal{E}}{\partial_r \langle q \rangle} + \langle v_{E\times B} \rangle\right] = 0}. \tag{29}$$

This indicates that **zonal mode and waves are non-linearly coupled**. If there is no particle flux, spreading, or other dissipations, **the relaxation wave is what drives the zonal flow**—one can accelerate the zonal flow *only if* change the WMD (i.e. change in fluctuation/turbulence intensity). This is also called the **non-acceleration theorem**[3]. Hence, zonal flow can be 'saturated' by zonal flow/fluctuation interaction such as:

- Collisional viscosity ν can damp the turbulence/fluctuation intensity. And since zonal flow is fed on turbulence/fluctuation intensity (predatory-prey model), zonal flow will be also damped.
- The system can reach a saturate state but not steady, so we have an energy oscillation called **Limit Cycle Oscillation**. This is similar to the predator-prey model but in different limits.
- Density corrugation can saturate zonal flow.
- **Turbulent viscosity** ν_{turb} can saturate the zonal flow when the system becomes unstable due to the shear flow instability. And this can be also viewed as a predator-prey model.

Homework: Derive the C-D theorem for ***forced*** *Charney equation. Compare it to H-W case.*

[2]Recall the wave momentum density $k\dfrac{\partial \epsilon}{\partial \omega}|E|^2$, where $\dfrac{\partial \epsilon}{\partial \omega}|E|^2$ is wave action density (N). And this pseudo-momentum density is rewritten in DW variables from the wave momentum density.

[3]Note that $\langle \delta f^2 \rangle$ sometimes related to 'entropy' and hence one can also have the 'entropy balance theorem'.

3 Reduced MHD

The simplest model for electromagnetics is reduced MHD. The key to the reduced MHD is the timescales. The key timescale ordering for H-W and H-M model is small Rossby number $Ro < 1$, i.e. $\omega < \Omega_i$(gyrofrequency, or ion cyclotron frequency). In MHD, there are 7 modes—$\mathbf{v}$ has three modes, $\mathbf{B}$ has two ($\nabla \cdot \mathbf{B} = 0$), 1 from pressure p and ρ respectively. 3 modes is interesting (the fast, medium, and slow mode), 1 is entropy mode ($\omega_{\text{en}} = 0$):

- Fast—Magnetosonic mode: $\omega_{MS}^2 = k_\perp^2(v_{A,\perp}^2 + C_s^2)$
- Intermediate—Shear Alfvén mode: $\omega_{SA}^2 = k_\parallel^2 v_A^2$ (supported by reduced MHD model)
- Slow—acoustic (parallel) mode: $\omega_{AC}^2 = k_\parallel^2 C_s^2$.

In this course (i.e. fusion tokamak), $\beta \equiv P_{\text{thermal}}/P_{\text{mag}}$ is small ($\simeq 0.05$), so $C_s < v_A$. In space, β can be large. The *legist of reduced MHD is to say we are considering a system with timescales that live longer than the magnetosonic wave(fast mode)*—eliminate the magnetosonic time scale ($\omega \ll \omega_{MS}$). In magnetosonic wave, the system compress the **stiff** magnetic fields in perpendicular direction. And the stiffness leads to high frequency. Reduced MHD eliminate these high frequency by introduce a *comparatively* strong straight magnetic field B_0 in a direction (hereafter we set the strong B_0 in z-direction). The *comparatively strong* means that the magnetic curvature of B_0 (R_{c,B_0}) is larger thang the length scale we consider ($k > 1/R_{\text{c},B_0}$). The strong field also means that $\rho v^2 \simeq p \ll B_{0,z}^2/8\pi$, and it leads to **quasi-2D** system where the motion is *strongly* anisotropic and all small-scale motions are generated in $\perp$ direction *only*.

3.1 Derivation of reduced MHD

One if the most famous reference for studying reduced MHD is Strauss (1976). We write down the Euler equation with $J \times B$ force

$$\rho \frac{\partial}{\partial t}\mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} = -\nabla p + \mathbf{J} \times \mathbf{B} = -\nabla\left(p + \underbrace{\frac{B^2}{2\mu_0}}_{\substack{\text{mag. pressure}\\ \text{force}}}\right) + \underbrace{\frac{(\mathbf{B}\cdot\nabla)\mathbf{B}}{\mu_0}}_{\substack{\text{mag. curvature}\\ \text{force}}} \tag{30}$$

Taking $\nabla\cdot$ of above equation and assuming incompressibility in perpendicular dierction ($\nabla_\perp \mathbf{v} = 0$), we have

$$0 = -\nabla^2\left(\frac{p}{\rho} + \frac{B^2}{2\mu_0 \rho}\right) + 0. \tag{31}$$

From above we obtain

$$\boxed{\delta p \simeq -\frac{\mathbf{B}\cdot\delta\mathbf{B}}{\mu_0}}. \tag{32}$$

This is the **perturbed pressure balance** in reduced MHD. To sumerize, in reduced MHD, we have

- consider timescales longer than the magnetosonic wave timescale $\tau \gg \tau_{MS}$,
- flow has incompressibility $\nabla\cdot\mathbf{v} = 0$ forced by strong $B_{0,z}$ (in fusion plasma we are interested in $\nabla_\perp \cdot \mathbf{v}_\perp \simeq 0$ and allows parallel sonic component), and
- perturbed pressure balance $\delta p \simeq -\dfrac{\mathbf{B}\cdot\delta\mathbf{B}}{\mu_0}$.

Because of the properties that eliminates fast mode and that its incompressibility, the reduced MHD supports only the **shear Alfvén wave** (linear wave). One can star reduced MHD with

- (a) Continuity equation of charge $\frac{\partial}{\partial t}\rho + \nabla\cdot\mathbf{J} = 0$ with charge conserve along the trajectory leads to $\nabla\cdot\mathbf{J} = \nabla\cdot(\mathbf{J}_{\parallel} + \mathbf{J}_{\perp}) = 0$ (i.e. plasma is **quasi-neutral**).

- (b) Electrical potential (ϕ) that defines velocity fields such that $E\times B$ drift velocity $\mathbf{v}_{E\times B} \equiv \hat{z}\times\nabla\phi/B_z$.

- (c) Lorentz gauge $\mathbf{E} = -\nabla\phi - \frac{\partial}{\partial t}\mathbf{A}$ and Ohm's Law[4] $E_{\parallel} = \eta J_{\parallel} = -\eta\nabla^2 A_{\parallel}$.

We apply an operator $(\nabla\times)\cdot\hat{z}$ on the Euler equation (see Eq.30) and obtain

$$\boxed{\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\nabla^2\phi = \cancelto{0}{-\nabla\times\frac{\nabla p}{\rho}} - \frac{\mathbf{B}\cdot\nabla(\nabla^2 A_{\parallel})}{\mu_0\rho} + \nu\nabla^2(\nabla^2\phi),} \tag{33}$$

where $\mathbf{B}\cdot\nabla \equiv B_0\partial_z + \delta B_{\perp}\nabla_{\perp}$. The first term on RHS $-\frac{\mathbf{B}\cdot\nabla(\nabla^2 A_{\parallel})}{\mu_0\rho}$ is from $\mathbf{J}\times\mathbf{B}$ torque and cause *non-linearity* in $\perp$ direction. And from Ohm's Law $\mathbf{E} + \mathbf{v}\times\mathbf{B} = \mu_0\eta\mathbf{J}$, we have induction equation

$$\boxed{\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\right)A_{\parallel} = \mathbf{B}\cdot\nabla\phi + \eta\nabla^2 A_{\parallel}}. \tag{34}$$

Note that $A_{\parallel}$ can be also viewed as 'magnetic flux'. These equation are reduced MHD equations—the **2 scalar evolution equations**. These equations are derived based on facts that p and ρ are unchanged in $\perp$ direction, and δB and δv are in $\perp$-plane, so we can reduce B and v to the gradient of scalar across z. All the non-linear term is in perpendicular direction ($\perp$), while all the linear terms are in parallel direction (z). Hence, a complicated problem can be decomposed into *2D dynamics* (on $\perp$-plane) plus *Shear Alfvén wave* (z- direction) problem. If we turn off lowest order term $B_0\partial_z \to 0$, this reduced MHD will become a rigorous 2D MHD equations. Note that there are 2 ways to derive reduced MHD from Boltzmann equation (see Fig.4)—via different orders of strong field approximation and fluid approximation.

Because of strong $B_{o,z}$, the ordering of reduced MHD (quasi-2D) is

$$\begin{aligned} B_{0,z} &\sim \nabla_{\perp} \sim \mathcal{O}(1) \\ B_{\perp} &\sim \nabla_z \sim \mathcal{O}(\epsilon) \\ \partial_t &\sim v_{\perp}\cdot\nabla_{\perp} \sim \mathcal{O}(\epsilon) \end{aligned} \tag{35}$$

And we take

$$\rho \sim \mathcal{O}(1) \text{ for } \nabla\cdot\mathbf{v} = 0, \tag{36}$$

$$\delta p \sim v_{\perp}^2 \sim B_{\perp}^2 \sim \mathcal{O}(\epsilon^2) \text{ for partition of energy, and} \tag{37}$$

$$\delta B_z \sim \mathcal{O}(\epsilon^2) \text{ for the pressure balance in Eq. 32 } \delta p \simeq -\frac{\mathbf{B}_0\cdot\delta\mathbf{B}_z}{\mu_0} \tag{38}$$

And since $\delta B_z = (\nabla_{\perp}\times A_{\perp})\cdot\hat{z}$ and δB_z is $\sim \mathcal{O}(\epsilon^2)$, we can neglect the inductive electric fields in perpendicular direction $\partial_t A_{\perp} \sim \mathcal{O}(\epsilon^3)$—this is the **electrostatic condition**. Hence, we have Lorentz gauge (if $\eta \to 0$) reduce to $E_{\perp} = -\nabla_{\perp}\phi - \cancelto{0}{\frac{\partial}{\partial t}A_{\perp}} = -(\mathbf{v}\times\mathbf{B})_{\perp}$. So we have

$$v_{\perp} = \frac{\hat{z}\times\nabla\phi}{B_0}, \tag{39}$$

[4]Note that if we consider the 'drift-Alfvén wave', we could consider the $\nabla_{\parallel}p$ in Ohm's Law balance. But usually $v_A \gg \rho C_s/L_n$, hence we ignore the DW effect.

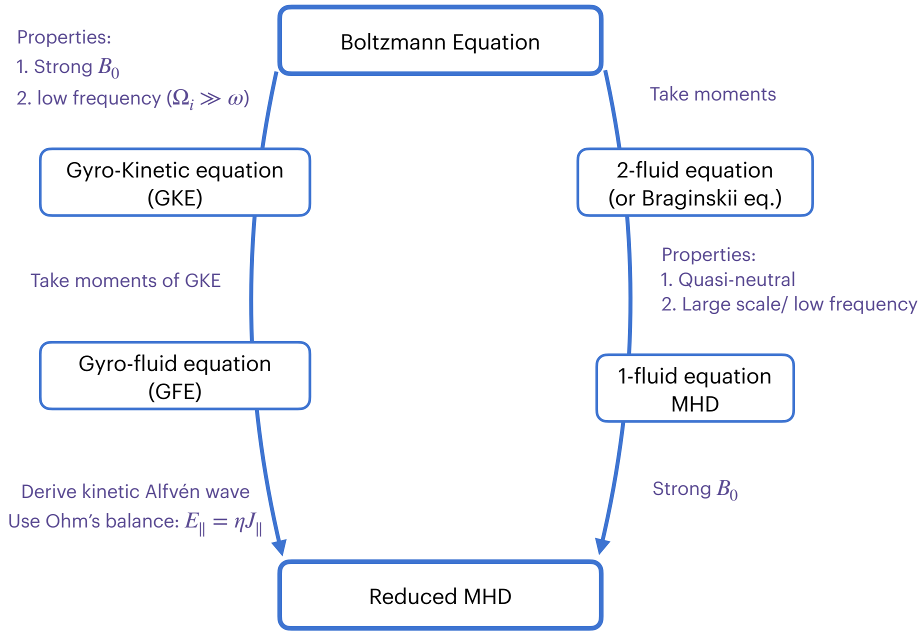

Figure 4: Two ways to derive RMHD from Boltzmann equation.

indicating that motion in perpendicular direction is $E \times B$ flow.

3.2 Conservation Law of reduced MHD

From Sec. 3.1, we derive that the physics of reduced MHD can be decomposed into *2D MHD* (incompressible) plus *Shear Alfvén* mechanism. So, one of conservation law is from incompressible 2D MHD—**total energy**

$$E_{\text{tot}} = \int d^3x \left(\frac{(\nabla\phi)^2}{2} + \frac{(\nabla A_\parallel)^2}{2} \right). \tag{40}$$

The total energy is consist of the kinetic energy part and the magnetic energy part.

The other conservation law is the **magnetic helicity** $\mathcal{H} \equiv \int d^3x \mathbf{A} \cdot \mathbf{B}$:

$$\mathcal{H} = \int d^3x A_\parallel \cdot B_\parallel. \tag{41}$$

The degenerated form of magnetic helicity in 2D MHD is $\mathcal{H}_{2D} = \int d^2x A^2 =$ conserved. But this term is not conserved—in 3D induced EMF term $B_0 \cdot \partial_z \phi \neq 0$.

The third conserved quantity is the **Cross Helicity** ($\mathcal{H}_c = \int d^3x \mathbf{v}\mathbf{B}$)

$$\mathcal{H}_c = \int d^3x \mathbf{v} \cdot \mathbf{B} = \int d^3x (\nabla\phi \cdot \nabla A_\parallel). \tag{42}$$

This equation is related to Poynting flux ($\mathbf{S}_p = \mathbf{E} \times \mathbf{B}/\mu_0$), and Alfvén wave population (2-direction propagation Alfvén waves) imbalance. All these conservation quantities above is conserved to magnetic and fluid dissipation.

3.3 Extensions

One of the extensions for reduced MHD is 4-filed equation, which has several versions

- Drake & Antonsen Jr (1984): ϕ, $A_{\|}$, n, T. This model is good foe drift-tearing mode (microscopic instability driven by the current gradient).
- Hasegawa (1981): ϕ, $A_{\|}$, $v_{\|}$, T_i. Good for ITG mode.

One can get the four-field theory if is familiar with reduced MHD and H-W equations.

> *Homework: Derive drift-Alfvén model starting by combining reduced MHD and H-W model.*

4 Zonal flow and DW

4.1 ZF in Jupiter Atmosphere

What generate zonal mode? The answer is **three waves interaction** (related to **Fermi's golden rule**). The production of zonal flow involve the *wave collision rate* ($C_{(N)}$) in **wave kinetic equation** (WKE) such that

$$C_{(N)} = \int d^3k' \Big(\cdots \Big) \delta(k^{"} - k - k^{'}) \delta(\omega_{k^{"}} - \omega_k - \omega_{k'}) \tag{43}$$

When there is no frequency mismatch $\omega_{k^{"}} - \omega_k - \omega_{k'} \simeq 0$, we have three wave resonance and can have zonal mode production.

Zonal flow can be found in fusion device or in astronomical objects such as Jupiter atmosphere and solar tachocline. Here we take Jupiter atmosphere as example. On the Jupiter atmosphere, Rossby energy dissipation emitted out meridionaly and momentum is transport *actively* from the equator that steepen the toroidal velocity gradient (see Fig.5). This can be illustrated by deriving group and phase velocity of Rossby wave frequency

$$\omega_{\text{Rossby},k} = -\frac{\beta k_x}{k_y^2}, \tag{44}$$

where x is toroidal (zonal) direction and y is in latitudinal direction. So the group velocity is

$$v_{g,y} = 2\beta \frac{k_x k_y}{(k_\perp^4)} \propto k_x k_y. \tag{45}$$

And we know that the latitudinal momentum transport (i.e. Reynolds stress) is $\langle \widetilde{v}_x \widetilde{v}_y \rangle \propto -k_x k_y$. Hence we have

$$\text{sign}(v_{g,y}) \cdot \text{sign}(\langle \widetilde{v}_x \widetilde{v}_y \rangle) < 0, \tag{46}$$

which indicates that Rossby wave is a *backward wave*—momentum transport into the zonal band and energy transport out.

> *Kushner et al. (2001): ... the central result that rapidly rating flow, when stirred in a local region, will converge angular momentum into this region.*

The momentum is deposit to the stirred region and form a zonal band (see Fig5). The direction of zonal flow depend on

- eastward in source region,
- westward in sink region,

- the rotation of planet $\beta > 0$, and
- negative diffusion phenomenon.

The phenomenon that momentum is transport to the up-gradient is the phenomenon of **negative diffusion**. In biology, the cell is supported by active transport across the membrane, and that *cost energy*, i.e. Adenosine Triphosphate (ATP). Because of the energy consumption for the negative diffusion, the entropy will increase globally, while the negative diffusion steepening the velocity gradient reduces entropy *locally*.

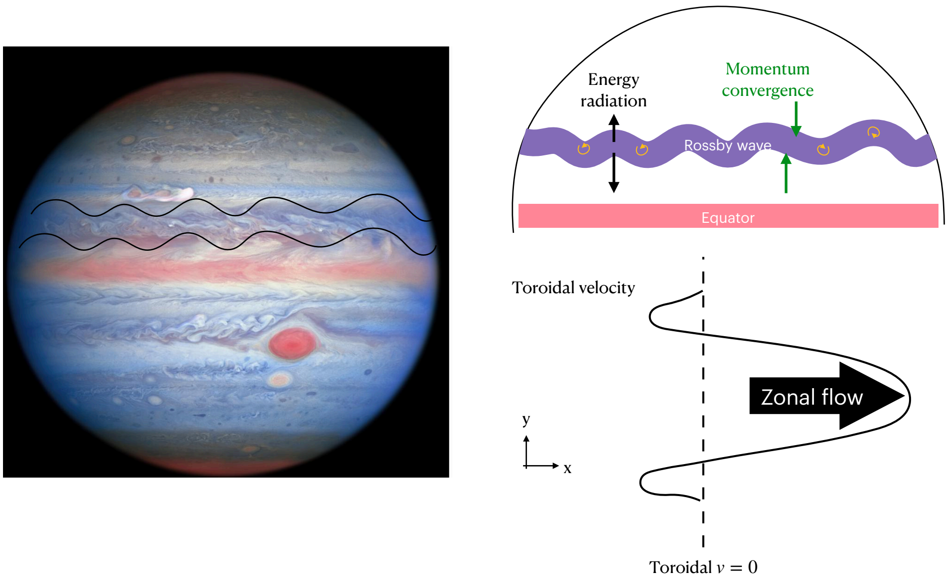

Figure 5: Left:Jupiter atmosphere. Right Top: cartoon of Rossby wave illustrated in purple band. Yellow swirls indicate stirring sources of Rossby wave. Right bottom: the toroidal (x-direction) velocity. There is opposite flow directions in north and south part of zonal flow for momentum deficit because of transport to the stirred region (momentum conservation).

4.2 ZF in Tokamak

Zonal flow in tokamak is

- Azimuthally symmetric: $n = 0$ potential mode of $E \times B$ flow (with possible sideband);
- $m = 0$

And has properties:

- produce no transport since $n = 0$
- has minimal inertia (Hasegawa, 1981; Sagdeev et al., 1978) for $k_\perp^2 \rho_s^2 < 1$. This means zonal mode are easy to excite.
- has minimal damping—no Landau damping where usually involve modes (van Kampen) that has resonant denominator $\propto \frac{1}{\omega - kv}$. Since $\omega \to 0$ and $k_\parallel \to 0$, there is no Landau damping. Hence, it is easy to put energy into zonal flows—once energy is in, it is weakly damped.
- ZF can be damped by collisional friction (see Sec.2.3), and other stability issues (nonlinear fluid damping) .

From these properties, we know that zonal flow is a natural predator that is easy to feed off and can retain energy released by gradient-driven micro-turbulence. Zonal flow can related to polarization flux (or PV transport) such that

$$\frac{\partial}{\partial_t}\langle v_{E\times B}\rangle = \langle \widetilde{v}_r \nabla_\perp^2 \widetilde{\phi}\rangle, \tag{47}$$

through Taylor identity (Taylor, 1915)

$$\langle \widetilde{v}_r \nabla_\perp^2 \widetilde{\phi}\rangle = \frac{\partial}{\partial r}\langle \widetilde{v}_r \widetilde{v}_\theta\rangle, \tag{48}$$

which relate flux of vorticity to the $E \times B$ Reynolds force.

4.3 Zonal Flow/DW interaction

In this section, we will investigate the shear flow formation and its interaction with turbulence. There are two ways to investigate: one is from real space study and another is from j-space coupling.

In real space, we know that zonal flows produce shears and shear can distort eddies as discussed in Sec.2 Fig.6. The shear decorrelation rate can be calculated as $1/\tau_c = k_x^2 D_\perp \to (k_y^2 \partial_x \langle v_{E\times B}\rangle D_\perp)^{1/3}$. This means we don't need too much diffusion to trigger the shear flow (Hahm, Burrell 1994). Shearing can affect wave-particle resonance—by producing shear 'differential' doppler shift such that

$$\omega - k_\parallel v_\parallel \to \omega - k_\parallel v_\parallel - k_y \frac{\partial}{\partial x} v_{E\times B}(r - r_0). \tag{49}$$

That create differential rotation. What happen when the shearing reaches saturation? How shear modify the growth? A crude simple reasoning is to compare shearing rate v.s. instability growth rate. When shear can *rip eddies up before they grow, we have shearing saturation*. In real space, turbulence drives Reynolds stress. Reynolds stress drives shear flow via momentum transport. And in turn, shear flows control the turbulence, and drives Reynolds stress again. This is the eddy self-tilting feedback loop (see Fig.6). **So, turbulence (wave) drive shear flow (zonal mode) via momentum transport, and shear flow controls turbulence and hence support the Reynolds stress.** Notice that this can be view as **predator prey model**—shear flow (zonal flow) is predator and turbulence (wave) is the prey (see Fig.8). And that turbulence and shear flow are highly nonlinear coupled—this is because wave and zonal mode are coupled nonlinearly as discussed in Sec.2.3 Eq.29.

In k-space, To study shear flow, we can start with **Eikonal theory** in k-space, because of the property that the scale of zonal flow is at larger scale than the waves and Eikonal theory is applicable. So, we can find the **wave kinetic equation** (WKE), which is in the form of Eikonal equation

$$\frac{\partial}{\partial t}N - \underbrace{\frac{\partial}{\partial k_r} D_k \frac{\partial}{\partial k_r}\langle N\rangle}_{\substack{\text{scattering in } k\text{-space} \\ \text{due to the eddy-tilting}}} = \underbrace{\gamma_k \langle N\rangle}_{\text{growth}} - \underbrace{\langle C(N)\rangle}_{\text{damping}} \tag{50}$$

where N is wave action density $N \equiv \dfrac{\text{wave Energy}}{\text{wave frequency}}$ and represents intensity of waves, $C(N)$ is the collision integral in Eq.43, and D_k is k-space diffusion. From Eikonal equation, we have the evolution of k_x in a shearing field

$$\frac{d}{dt}k_x = -\frac{\partial(\omega_0 + u_y k_y)}{\partial x} = -k_y \frac{\partial u_y}{\partial x} \tag{51}$$

This is the eddy tilting feedback loop aligning wave vector components. Once $\langle k_x k_y \rangle$, flow evolution occurs due to momentum transport. Then, flow shear amplification further amplifies the Reynolds stress, etc (see Fig.6). The diffusion in k-space is

$$D_k = \sum_k k_y^2 |\frac{\partial}{\partial x} \widetilde{v}_{E\times B}|^2 \tau_k, \tag{52}$$

which is the **induced diffusion**.

To summarize, zonal flow (shear flows) and waves turbulence forms an self-tilting feedback loop that can be described as the predator prey model. The intensity of turbulence (i.e. population of prey) depends on the intensity of zonal flow (i.e. predator). The collisional friction or nonlinear fluid damping can control the zonal flow and turbulence can grow—just like we allow *hunting on big cats then antelopes can grow* in the predator prey model. So the real control of wave turbulence in the eddy-tilting feedback loop is flow damping, either collisional or nonlinear.

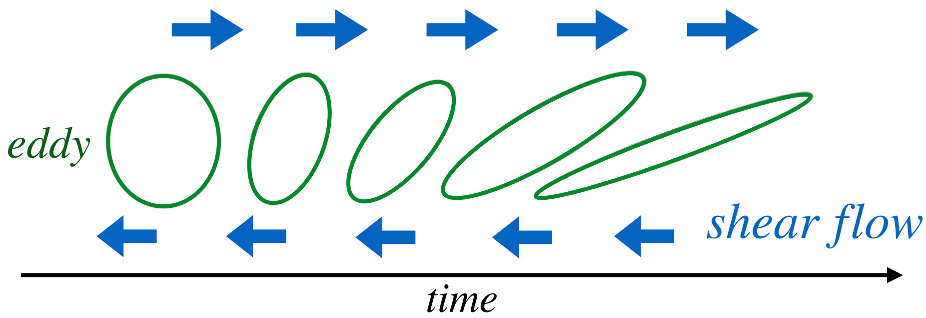

Figure 6: Shear-eddy tilting feedback loop. The $E \times B$ shear generates the $\langle k_x k_y \rangle$ correlation and hence support the non-zero Reynolds stress. The Reynold stress, in turn, modifies the shear via momentum transport. Hence, the shear flow reinforce the self-tilting.

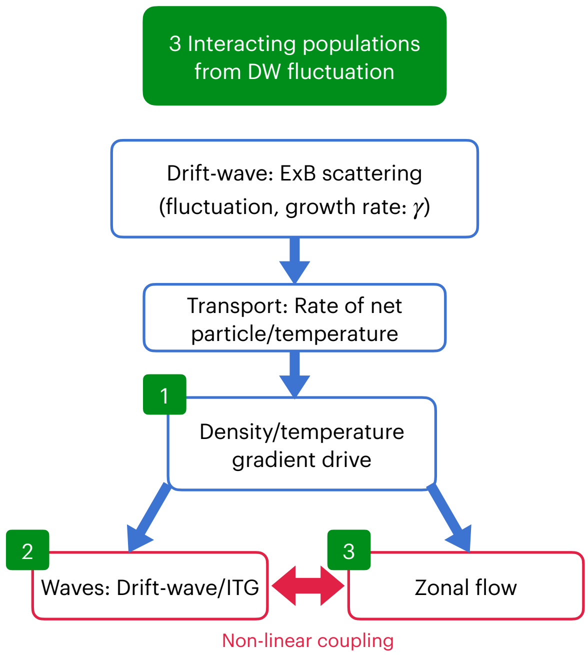

Figure 7: There are thee interaction populations from a single fluctuation intensity.

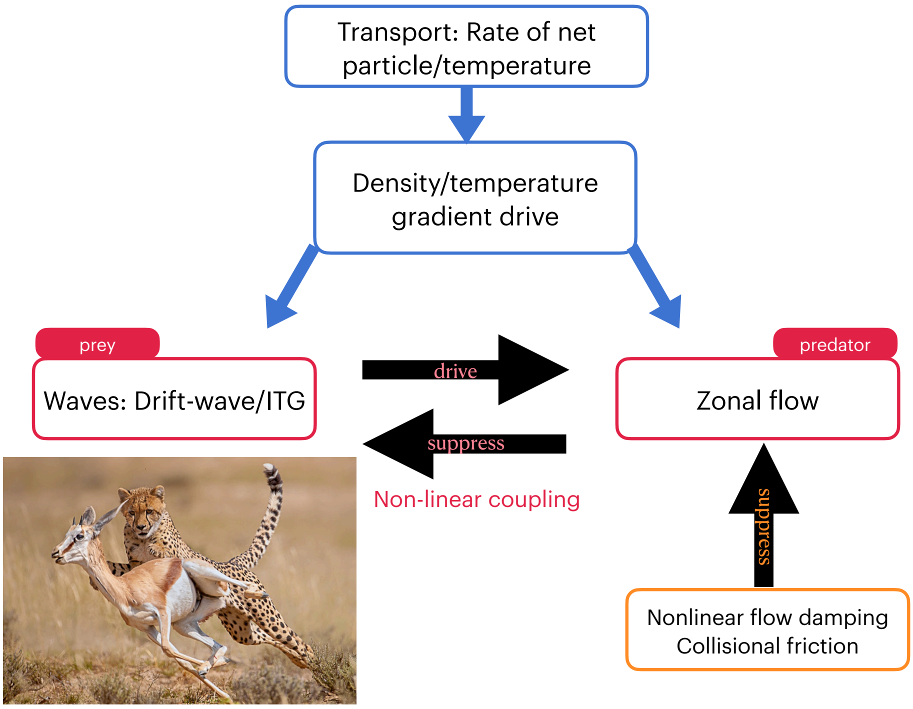

Figure 8: The predator prey model.

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